— Exercises —

1. **Approximation and Taylor's formula.** Let f be the function defined on $\mathbb{R}^2$ by $f(x,y) = exp(sin(x)cos(y))$. Show that the error made in computing $f(3.16, 0.02)$ by approximating $f(x,y)$ by $f(\pi,0) + (3.16-\pi)\frac{\partial f}{\partial x}(\pi,0) + 0.02\frac{\partial f}{\partial y}(\pi,0)$ is smaller than $\frac{16e}{10^4}$.

2. **Extrema of functions on $\mathbb{R}^2$.** Find local extrema and their types for the following functions $\mathbb{R}^2 \to \mathbb{R}$.

   (a) $(x,y) \mapsto x + y + x^2 + y^2 + xy$

   (b) $(x,y) \mapsto y^2 - x^3$

3. Let f be a $\mathcal{C}^2$ function defined on $\mathbb{R}$ such that $f(0) = 0$ and that $f'(0) \neq 0$. We put $F(x,y) = f(x)f(y)$ for every $(x,y) \in \mathbb{R}^2$. Is $(0,0)$ a critical point of F? Is it a local (global) extremum?

4. **Laplacian and Hessian matrix.** Let f be a $\mathcal{C}^2$ function defined on an open subset of $\mathbb{R}^n$.

   (a) Check that the Laplacian of f at a point x is the trace of the Hessian matrix at x.

   (b) We assume that x is a local minimum (resp. local maximum) of f, show that $\Delta f(x) \geq 0$ (resp. $\Delta f(x) \leq 0$).

5. **Functions with non-open domains.** Consider a function $f : [0,1] \to \mathbb{R}$. What is the meaning of "x is a local max of f"? Find such an f showing that the implication "local extremum $\Rightarrow$ critical point" is not true for a function whose domain is not open.

— Problems —

6. **Global extrema.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function.

   (a) Give an example of such an f having a local min that is not a global min.

   (b) Show that if $f(\mathbf{x}) \xrightarrow{\|\mathbf{x}\|\to\infty} \infty$, then f has a global min.

7. **Product of sin in a triangle.** Let T be a triangle whose angles are x, y and z. We want to find the points $(x,y,z) \in [0,\pi]^3$ where the product $P(x,y,z) = sin(x)sin(y)sin(z)$ has its global maximum.

   (a) Find a function g defined on a compact set $K \subset \mathbb{R}^2$ such that $g(x,y) = P(x,y,z)$ for every (x,y,z) corresponding to a triangle. Justify that g admits a global maximum.

   (b) Is the global maximum of g obtained on the boundary of K?

   (c) Compute the critical points of $g_{|\mathring{K}}$. Conclude.

8. **Maximum principle.** Let U be an open set in $\mathbb{R}^n$ and f a $\mathcal{C}^2$ function on U.

   (a) Let K be a compact set included in U and $\mathring{K}$ its interior. Show that $f_{|\partial K}$ admits a maximum at some $a \in \partial K$.

   (b) We assume that $\Delta f > 0$ in any point of $\mathring{K}$. Show that $f(a) = \max_K f$ and $f_{|\mathring{K}} < \max_{\partial K} f$.

   (c) Show that, if f is harmonic on $\mathring{K}$, then $\min_{\partial K} f \leq f_{|K} \leq \max_{\partial K} f$. *Hint: Define* $f_\epsilon(x) = f(x) + \epsilon||x||^2$ *and apply* (a).

   (d) Show that, if f is harmonic and constant on ∂K, then f is constant on K.